

Exotic Differential Operators on Complex Minimal Nilpotent Orbits

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Abstract

The orbit \mathcal{O} of highest weight vectors in a complex simple Lie algebra \mathfrak{g} is the smallest non-zero adjoint orbit that is conical, i.e., stable under the Euler dilation action of \mathbb{C}^* on \mathfrak{g} . \mathcal{O} is a smooth quasi-affine variety and its closure $\text{Cl}(\mathcal{O})$ is a singular quadratic cone in \mathfrak{g} . The varieties \mathcal{O} and $\text{Cl}(\mathcal{O})$ have the same algebra of regular functions, i.e. $R(\mathcal{O}) = R(\text{Cl}(\mathcal{O}))$, and hence the same algebra of algebraic differential operators, i.e., $\mathcal{D}(\mathcal{O}) = \mathcal{D}(\text{Cl}(\mathcal{O}))$.

The content and structure of $\mathcal{D}(\mathcal{O})$ is unknown, except in the case where $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. The Euler action defines natural \mathfrak{g} -invariant algebra gradings $R(\mathcal{O}) = \bigoplus_{p \in \mathbb{Z}_+} R_p(\mathcal{O})$ and $\mathcal{D}(\mathcal{O}) = \bigoplus_{p \in \mathbb{Z}} \mathcal{D}_p(\mathcal{O})$ so that $D(R_q(\mathcal{O})) \subset R_{q+p}(\mathcal{O})$ if $D \in \mathcal{D}_p(\mathcal{O})$.

We construct, for \mathfrak{g} classical and different from $\mathfrak{sp}(2n, \mathbb{C})$, a subalgebra \mathcal{A} inside $\mathcal{D}(\mathcal{O})$ which, like $R(\mathcal{O})$, is \mathfrak{g} -stable, graded and maximal commutative. Unlike $R(\mathcal{O})$, which lives in non-negative degrees, our algebra \mathcal{A} lives in non-positive degrees so that $\mathcal{A} = \bigoplus_{p \in \mathbb{Z}_+} \mathcal{A}_{-p}$. The intersection $\mathcal{A} \cap R(\mathcal{O})$ is just the constants. The space \mathcal{A}_{-1} is finite-dimensional, transforms as the adjoint representation and generates \mathcal{A} as an algebra. Moreover we construct a \mathfrak{g} -equivariant graded algebra isomorphism $R(\mathcal{O}) \rightarrow \mathcal{A}$, $f \mapsto D_f$. The differential operators in \mathcal{A}_{-1} have order 4.

The operators in \mathcal{A} are “exotic” in that they lie outside the realm of familiar differential operators. Although their existence is predicted (as a conjecture) by the quantization program for nilpotent orbits of the second author, our actual construction, by quantizing “exotic” symbols we obtained previously, is quite subtle.

We will use the operators in \mathcal{A}_{-1} to quantize \mathcal{O} in a subsequent paper and to construct an algebraic star product on $R(\mathcal{O})$. To pave the way for the quantization, we show that the formula $(f|g) = (D_{\bar{f}}f)(0)$ defines a positive-definite Hermitian inner product on $R(\mathcal{O})$.

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1 Introduction

The nature of the algebra $\mathcal{D}(X)$ of differential operators on a complex quasi-projective algebraic variety X is rather mysterious. There is only one rather general result: if X is both smooth and affine, then (i) $\mathcal{D}(X)$ is generated by the functions and vector fields on X , and (ii) every function on T^*X which is a homogeneous polynomial on the fibers of $T^*X \rightarrow X$ is the principal symbol of a differential operator. As soon as X becomes non-smooth or non-affine, the generation quickly fails and the surjectivity of the symbol map is not well understood.

Let X be a homogeneous space of a reductive complex algebraic group K . Then X is a quasi-projective complex algebraic manifold. Differentiation of the K -action gives an infinitesimal action of the Lie algebra \mathfrak{k} on X by vector fields. This gives rise to a complex algebra homomorphism

$$\mathcal{U}(\mathfrak{k}) \rightarrow \mathcal{D}(X) \tag{1}$$

where $\mathcal{U}(\mathfrak{k})$ is the universal enveloping algebra of \mathfrak{k} .

Suppose X is compact, i.e., projective. Then W. Borho and J.-L. Brylinski ([Bo-Br]) proved that the homomorphism (1) is surjective. They also computed its kernel which is a primitive ideal in $\mathcal{U}(\mathfrak{k})$. Here X is a (generalized) flag variety K/Q . These spaces X occur as the projectivized orbits of highest weight vectors in finite-dimensional irreducible representations of K . We note that K admits only finitely many compact homogeneous spaces. These orbits play a key role in algebraic geometry, symplectic geometry and representation theory.

There are natural non-compact analogues to the flag varieties. These homogeneous spaces of K arise in the following way. Suppose that K sits inside a larger reductive complex algebraic group G as (the identity component of) a spherical subgroup. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding complex Cartan decomposition where $\mathfrak{k} = \text{Lie } K$ and $\mathfrak{g} = \text{Lie } G$. Then the K -orbits in \mathfrak{p} are quasi-affine homogeneous spaces X of K (and so non-compact except if X is a point). The choice $G = K \times K$ gives rise to all the adjoint orbits of K .

Among all K -orbits in \mathfrak{p} , there are only finitely many orbits Y which are stable under the Euler dilation action of \mathbb{C}^* on \mathfrak{p} . These orbits are exactly the ones consisting of nilpotent elements of \mathfrak{g} ; accordingly they are called “nilpotent orbits”. There is a rich literature on the geometry of these orbits Y , especially with respect to the quantization of the corresponding (see [Se]) real nilpotent orbits. The most recent development here is the Kaehler structure on real nilpotent orbits arising from the work of Kronheimer ([Kr]) and Vergne ([Ve]). See also [B1], [B2], [B3].

The Euler action on Y provides K -invariant algebra gradings $R(Y) = \bigoplus_{p \in \mathbb{Z}} R_p(Y)$ and $\mathcal{D}(Y) = \bigoplus_{p \in \mathbb{Z}} \mathcal{D}_p(Y)$ where $R_p(Y)$ is the subspace of Euler homogeneous func-

tions of degree p and

$$\mathcal{D}_p(Y) = \{D \in \mathcal{D}(Y) \mid D(R_q(Y)) \subset R_{q+p}(Y) \text{ for all } q \in \mathbb{Z}\} \quad (2)$$

We say that differential operators in $\mathcal{D}_p(Y)$ are *Euler homogeneous of degree p* . The vector fields of the infinitesimal \mathfrak{k} -action are Euler homogeneous of degree 0 since the K -action commutes with the Euler action.

$\mathcal{D}(Y)$ is not, in general, generated by the global functions and vector fields on Y . We can see this already in the simplest case, namely the case where $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, $\mathfrak{k} = \mathfrak{gl}(n, \mathbb{C})$ and Y is a minimal (non-zero) orbit in \mathfrak{p} . (So Y is the orbit of highest weight vectors in \mathfrak{p}^+ or \mathfrak{p}^- .) Then Y is isomorphic to the quotient $(\mathbb{C}^n - \{0\})/\mathbb{Z}_2$ where \mathbb{Z}_2 acts by ± 1 . It is easy to see that $\mathcal{D}(Y)$ is the even part of the Weyl algebra, i.e.,

$$\mathcal{D}(Y) = \mathbb{C}[x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]^{\mathbb{Z}_2} = \mathbb{C}[x_i x_j, x_i \frac{\partial}{\partial x_j}, \frac{\partial^2}{\partial x_i \partial x_j}] \quad (3)$$

Clearly the subalgebra generated by the functions and vector fields is $\mathbb{C}[x_i x_j, x_i \frac{\partial}{\partial x_j}]$. This subalgebra lives in non-negative degrees, and so does not contain the operators $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. The operators D_{ij} generate a maximal commutative subalgebra \mathcal{A} of $\mathcal{D}(Y)$ which lives in non-positive degrees.

In most cases beyond the example above, it is a hard open problem to determine $\mathcal{D}(Y)$ and to construct any “new operators” in it beyond the known operators, i.e., those generated by functions and by the vector fields of the \mathfrak{g} -action. See Section 5 for a few known examples where highly non-obvious “new” operators in $\mathcal{D}(Y)$ have been constructed.

We expect $\mathcal{D}(Y)$ to contain “new” differential operators for general nilpotent orbits Y (cf. [B3]). These will be “exotic” in the sense that they lie outside the realm of familiar differential operators.

In this paper, we study the case where Y is the complex minimal nilpotent orbit \mathcal{O} of a simple complex Lie group G . (This is the case where we embed K as a spherical subgroup of $K \times K$; then we rename K as G .) See [J] and [B-K1] for some results on the geometry and quantization of \mathcal{O} .

The space \mathcal{O} is just the orbit of highest weight vectors (or equivalently, of highest root vectors) in \mathfrak{g} . Accordingly \mathcal{O} has a number of very nice properties. The closure of \mathcal{O} is $\text{Cl}(\mathcal{O}) = \mathcal{O} \cup \{0\}$ and $\text{Cl}(\mathcal{O})$ is a quadratic cone in \mathfrak{g} , i.e., $\text{Cl}(\mathcal{O})$ is cut out (scheme-theoretically) by homogeneous quadratic polynomial functions on \mathfrak{g} . All regular functions on \mathcal{O} extend to $\text{Cl}(\mathcal{O})$. So \mathcal{O} and $\text{Cl}(\mathcal{O})$ have the same algebra of regular functions, i.e., $R(\mathcal{O}) = R(\text{Cl}(\mathcal{O}))$, and hence the same algebra of algebraic differential operators, i.e., $\mathcal{D}(\mathcal{O}) = \mathcal{D}(\text{Cl}(\mathcal{O}))$. Notice that \mathcal{O} “misses” being a smooth affine variety by just one point, namely the origin.

The algebra $R(\mathcal{O})$ is graded in non-negative degrees so that $R(\mathcal{O}) = \bigoplus_{p \in \mathbb{Z}_+} R_p(\mathcal{O})$. The subspace $R_1(\mathcal{O})$ generates $R(\mathcal{O})$ as an algebra and consists of the functions

f_x , $x \in \mathfrak{g}$, obtained by restricting the linear functions on \mathfrak{g} . So f_x is defined by $f_x(z) = (x, z)_{\mathfrak{g}}$ where $(\cdot, \cdot)_{\mathfrak{g}}$ is the (normalized) Killing form of \mathfrak{g} .

Differentiation of the G -action on \mathcal{O} gives an infinitesimal vector field action

$$\mathfrak{g} \rightarrow \mathfrak{Vect} \mathcal{O}, \quad x \mapsto \eta^x \quad (4)$$

These vector fields η^x are Euler homogeneous of degree 0. The subalgebra of $\mathcal{D}(\mathcal{O})$ generated by $R(\mathcal{O})$ and by the η^x is thus Euler graded in non-negative degrees.

We assume that \mathfrak{g} is classical in order to prove our results. We exclude the case $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, $n \geq 1$, as then $\mathcal{O} \simeq (\mathbb{C}^{2n} - \{0\})/\mathbb{Z}_2$ and so $\mathcal{D}(\mathcal{O})$ is just the even part of a Weyl algebra.

We construct inside $\mathcal{D}(\mathcal{O})$ a G -stable, graded, maximal commutative subalgebra \mathcal{A} isomorphic to $R(\mathcal{O})$. Unlike $R(\mathcal{O})$, which lives in non-negative degrees, our algebra \mathcal{A} lives in non-positive degrees so that

$$\mathcal{A} = \bigoplus_{p \in \mathbb{Z}_+} \mathcal{A}_{-p} \quad (5)$$

where $\mathcal{A}_{-p} = \mathcal{A} \cap \mathcal{D}_{-p}(\mathcal{O})$. The intersection $\mathcal{A} \cap R(\mathcal{O})$ is just $\mathcal{A}_0 = R_0(\mathcal{O}) = \mathbb{C}$. The space \mathcal{A}_{-1} is finite-dimensional, transforms as the adjoint representation and generates \mathcal{A} as an algebra. We construct a G -equivariant graded algebra isomorphism $R(\mathcal{O}) \rightarrow \mathcal{A}$, $f \mapsto D_f$.

The differential operators in \mathcal{A}_{-1} have order 4 and are “exotic” in the sense discussed above. We construct \mathcal{A}_{-1} by quantizing a space $r(\mathfrak{g}) \subset R(T^*\mathcal{O})$ of symbols which we obtained in [A-B2]. We next recall the context in which we found $r(\mathfrak{g})$.

In [A-B1], we constructed complex Lie algebra embedding

$$\tau : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathbb{C}(T^*\mathcal{O}) \quad (6)$$

where $\mathbb{C}(T^*\mathcal{O})$ is the complex Poisson algebra of rational functions on $T^*\mathcal{O}$. We did this in a more general context which includes the cases $Y = \mathcal{O}$.

The restriction of τ to the diagonal in $\mathfrak{g} \oplus \mathfrak{g}$ is the natural map

$$\Phi : \mathfrak{g} \rightarrow R(T^*\mathcal{O}), \quad x \mapsto \Phi^x \quad (7)$$

where Φ^x is the principal symbol of the vector field η^x . I.e., $\tau(x, x) = \Phi^x$. On the anti-diagonal τ is given by

$$\tau(x, -x) = f_x - g_x \quad (8)$$

where g_x , $x \in \mathfrak{g}$, are rational functions on $T^*\mathcal{O}$ defined on a G -invariant (Zariski) open algebraic submanifold $(T^*\mathcal{O})^{reg}$.

The functions g_x transform in the adjoint representation of G . Each function g_x is homogeneous of degree 2 on the fibers of $T^*\mathcal{O} \rightarrow \mathcal{O}$. Also g_x is homogeneous of degree -1 with respect to the canonical lift of the Euler scaling action on \mathcal{O} . The functions g_x , like the functions f_x , Poisson commute; i.e., $\{f_x, f_y\} = \{g_x, g_y\} = 0$ for all $x, y \in \mathfrak{g}$.

In [A-B2], we computed g_x for a restricted class of orbits Y which still includes $Y = \mathcal{O}$. Let $\lambda \in R(T^*\mathcal{O})$ be the principal symbol of the Euler vector field E on \mathcal{O} . We found that $(T^*\mathcal{O})^{reg}$ is the locus in $T^*(\mathcal{O})$ where λ is non-vanishing. We got the formula:

$$g_x = \frac{r_x}{\lambda^2} \quad (9)$$

where

$$r : \mathfrak{g} \rightarrow R(T^*\mathcal{O}), \quad x \mapsto r_x \quad (10)$$

is an explicit complex linear \mathfrak{g} -equivariant map.

Each function r_x is a homogeneous degree 4 polynomial on the fibers of $T^*(\mathcal{O}) \rightarrow \mathcal{O}$. Also r_x is homogeneous of degree -1 with respect to the canonical lift of the Euler action. The functions r_x again Poisson commute, i.e., $\{r_x, r_y\} = 0$.

We say that a function $\phi \in R(T^*X)$ which is homogeneous of degree d on the fibers of $T^*X \rightarrow X$ (and hence a polynomial function on the fibers) is an *order d symbol*. The order d symbol map sends $\mathcal{D}^d(X)$ into $R^d(T^*X)$ where $\mathcal{D}^d(X)$ is the space of differential operators of order at most d and $R^d(T^*X)$ is the space of order d symbols. We say that $\phi \in R^d(T^*X)$ *quantizes* into D if $D \in \mathcal{D}^d(X)$ has order d and the principal symbol of D is ϕ . Unless X is smooth and affine, there is no guarantee that a given symbol will quantize.

In this paper we quantize the symbols r_x into order 4 differential operators D_x on \mathcal{O} in a manner equivariant with respect to both the G -action and the Euler \mathbb{C}^* -action. We show that this equivariant quantization is unique. In our next paper [A-B3], we use these same operators D_x to quantize \mathcal{O} by quantizing the map (6). We obtain a star product on $R(\mathcal{O})$ given by “pseudo-differential” operators.

We construct the operators D_x by manufacturing a single operator $D_0 = D_{x_0} \in \mathcal{D}_{-1}^4(\mathcal{O})$ where $x_0 \in \mathfrak{g}$ is a lowest weight vector. Here $\mathcal{D}_p^d(\mathcal{O}) = \mathcal{D}^d(\mathcal{O}) \cap \mathcal{D}_p(\mathcal{O})$. We build D_0 so that its principal symbol is $r_0 = r_{x_0}$ and D_0 is a lowest weight vector for a copy of \mathfrak{g} in $\mathcal{D}_{-1}^4(\mathcal{O})$. Then all other operators D_x are simply obtained by taking (iterated) commutators of D_0 with the vector fields η^y .

To construct D_0 , we use our explicit formula for r_0 from [A-B2] to quantize r_0 into a differential operator on a (Zariski) open set \mathcal{O}^{reg} in \mathcal{O} . Our quantization method is a modification of Weyl quantization by symmetrization. We are forced to modify the quantization of a factor λ^2 appearing in a term of r_0 in order that the quantization

of r_0 is \mathfrak{g} -finite and hence extends to an operator on all of \mathcal{O} . See Subsection 3.3 for a detailed discussion of our strategy.

In essence, we are adding lower order correction terms to the true Weyl quantization of r_0 in order to obtain D_0 . The correction is uniquely determined, but its nature is mysterious to us.

This paper can largely be read independently of [A-B1] and [A-B2], as the symbols in fact only motivate the construction of the differential operators. Once we figure out the correct formula for D_0 , we give a self-contained proof that $D_0 \in \mathcal{D}_{-1}^4(\mathcal{O})$. The more abstract and general results we prove in Section 2 for differential operators on cones of highest weight vectors then give in particular the main properties of our operators D_x : (i) the operators D_x commute, (ii) the operators D_x generate a maximal commutative subalgebra of $\mathcal{D}(\mathcal{O})$, and (iii) $\overline{f_x}$ and D_x are adjoint operators on $R(\mathcal{O})$ with respect to a (unique) positive definite Hermitian inner product $(\cdot|\cdot)$ on $R(\mathcal{O})$ such that $(1|1) = 1$.

A basic question in quantization theory is whether one can set up a preferred quantization converting functions on T^*X which are polynomial on the fibers of $T^*X \rightarrow X$ into differential operators. This certainly involves quantizing symbols into operators which include “lower order” terms. This has been studied in a different context in [L-O] – see Section 5.

In Section 5 we compare our constructions with the results of Levasseur, Smith and Stafford on the Joseph ideal and on differential operators on classical rings of invariants, and with the constructions in [B-K2].

2 Differential operators on conical orbits

2.1 Differential operators on G -varieties

Let X be a complex algebraic quasi-projective variety. Let \mathcal{D}_X be the sheaf of algebraic differential operators on X . Then $\mathcal{D}(X) = \Gamma(X, \mathcal{D}_X)$ is the algebra of (global) differential operators on X . Let $\mathcal{D}^d(X) \subset \mathcal{D}(X)$ be the subspace of differential operators of order at most d . See e.g. [B-K2, Appendix] for some of the basic definitions and facts on differential operators.

Let $R(X)$ denote the algebra of regular functions on X . For any commutative complex algebra A , let $\mathfrak{D}(A)$ be the algebra of differential operators of A . We have a natural algebra homomorphism $\mathcal{D}(X) \rightarrow \mathfrak{D}(R(X))$ and this is an isomorphism if X is quasi-affine. We will be dealing primarily with differential operators on quasi-affine varieties.

Throughout the paper, we are working with complex quasi-projective algebraic varieties. The functions, vector fields, and differential operators we deal with are all regular, or algebraic, in the sense of algebraic geometry.

Our main emphasis is on working with smooth varieties, which are then complex algebraic manifolds and admit cotangent bundles. We often use the term “algebraic holomorphic” in place of “regular” to emphasize that the algebraic structure is an overlay to the underlying holomorphic structure.

Now suppose X is smooth. Let G be a reductive complex algebraic group. Assume G acts on X by an algebraic holomorphic group action. Then there are induced representations of G on $R(X)$ and on $\mathcal{D}(X)$ by (complex) algebra automorphisms. Both G -representations are locally finite and completely reducible.

We use the term *G -linear map* to mean a G -equivariant complex linear map of vector spaces. A vector space V is *G -irreducible* if V is an irreducible representation of G ; often V will be a G -stable space inside a larger representation. If V is G -irreducible and W is some G -representation, then we say that a non-zero G -linear map $V \rightarrow W$ gives a “copy of V inside W ”.

The differential of the G -action on X is the infinitesimal algebraic holomorphic vector field action

$$\mathfrak{g} \rightarrow \mathfrak{Vect} X, \quad x \mapsto \eta^x \quad (11)$$

where $\mathfrak{g} = \text{Lie}(G)$ and $\eta_z^x = \frac{d}{dt}|_{t=0}(\exp -tx) \cdot z$. These vector fields give a complex Lie algebra homomorphism

$$\mathfrak{g} \rightarrow \text{End } \mathcal{D}(X), \quad x \mapsto \text{ad } \eta^x \quad (12)$$

where $\text{ad } \eta^x(D) = [\eta^x, D]$. This representation of \mathfrak{g} on $\mathcal{D}(X)$ extends canonically to a representation of \mathfrak{g} on $\mathcal{D}(X')$ where X' is any Zariski open set in X .

Proposition 2.1.1 *Suppose G acts transitively on X . Let D be a differential operator on a Zariski open set in X . Then D extends to a differential operator on X if and only if D is \mathfrak{g} -finite.*

Proof: We have a filtration of \mathcal{D}_X by the sheaves \mathcal{D}_X^k of differential operators on X of order at most k . For each k , \mathcal{D}_X^k is the sheaf of sections of some complex vector bundle on X of finite rank. So the result follows by the same general lemma we used in [A-B2] to prove the corresponding statement for $R(T^*X)$. \square

We say that X is a G -cone if there is a (non-trivial) algebraic holomorphic group action of \mathbb{C}^* on X commuting with the G -action. The infinitesimal generator of the \mathbb{C}^* -action is the algebraic holomorphic *Euler vector field* E on X . The Euler action defines G -invariant algebra gradings

$$R(X) = \bigoplus_{p \in \mathbb{Z}} R_p(X) \quad \text{and} \quad \mathcal{D}(X) = \bigoplus_{p \in \mathbb{Z}} \mathcal{D}_p(X) \quad (13)$$

where

$$\begin{aligned} R_p(X) &= \{f \in R(X) \mid E(f) = pf\} \\ \mathcal{D}_p(X) &= \{D \in \mathcal{D}(X) \mid [E, D] = pD\} \end{aligned}$$

Then, if X is quasi-affine,

$$\mathcal{D}_p(X) = \{D \in \mathcal{D}(X) \mid D(R_q(X)) \subset R_{p+q}(X) \text{ for all } q \in \mathbb{Z}\} \quad (14)$$

The Euler grading of $\mathcal{D}(X)$ is compatible with the order filtration so that we get a grading of the space of differential operators of order at most d

$$\mathcal{D}^d = \bigoplus_{p \in \mathbb{Z}} \mathcal{D}_p^d \quad (15)$$

where $\mathcal{D}_p^d = \mathcal{D}^d \cap \mathcal{D}_p$.

2.2 Differential operators on the cone of highest weight vectors.

In this subsection, we assume that X is the G -orbit of highest weight vectors in a (non-trivial) finite-dimensional irreducible complex representation V of G . A (non-zero) vector v in V is called a *highest weight vector* if v is semi-invariant under some Borel subgroup B of G . (In Cartan-Weyl highest weight theory, one usually fixes a choice of B , and then there is, up to scaling, exactly one B -semi-invariant vector. The conjugacy of Borel subgroups implies that the set of all highest weight vectors is an orbit.)

X is a locally closed subvariety of V and is stable under the scaling action of \mathbb{C}^* on V . So X is a G -cone with Euler vector field E . In fact the quotient X/\mathbb{C}^* is the unique closed G -orbit in the projective space $\mathbb{P}(V)$. The quotient X/\mathbb{C}^* is often called the projectivization $\mathbb{P}(X)$ of X . The closure $\text{Cl}(X)$ of X in V is equal to $X \cup \{0\}$ and X is the unique non-zero minimal conical G -orbit in V .

The orbit X has several very nice properties ([V-P]):

- (i) the closure $\text{Cl}(X)$ is normal.
- (ii) the graded G -equivariant algebra homomorphism $R(\text{Cl}(X)) \rightarrow R(X)$ given by restriction of functions is an isomorphism.
- (iii) the algebra $R(X)$ is multiplicity-free as a G -representation and $R_p(X)$ is G -isomorphic to the p th Cartan power of V^* .

These properties of X can be derived using the Borel-Weil theorem on $\mathbb{P}(X)$.

If a G -irreducible vector space W carries the G -representation V_μ , then the p th Cartan power of W is the (unique) G -irreducible subspace $W^{\boxtimes p}$ of the p th symmetric power $S^p(W)$ which carries $V_{p\mu}$. Here V_ν denotes the finite-dimensional irreducible G -representation of highest weight ν . Geometrically, if W is the space of holomorphic sections of a G -homogeneous complex line bundle \mathcal{L} over $\mathbb{P}(X)$, then $W^{\boxtimes p}$ identifies with the space of holomorphic sections of $\mathcal{L}^{\otimes p}$.

The algebra $R(V)$ identifies with the symmetric algebra $S(V^*)$. Properties (ii) and (iii) say that the natural graded algebra homomorphism

$$\zeta : S(V^*) \rightarrow R(X) \quad (16)$$

defined by restriction of polynomial functions from V to X is surjective and the p th graded map $\zeta_p : S^p(V^*) \rightarrow R_p(X)$ induces a G -linear isomorphism $\zeta_p : (V^*)^{\boxtimes p} \rightarrow R_p(X)$.

Consequently, $R_p(X) = 0$ for $p < 0$ and

$$R_0(X) = \mathbb{C} \quad (17)$$

Also $R(X)$ is generated by $R_1(X)$. In degree 1, ζ gives the G -linear isomorphism

$$V^* \rightarrow R_1(X), \quad \alpha \mapsto f_\alpha \quad (18)$$

where $f_\alpha(w) = \alpha(w)$. Also property (ii) implies that

$$\mathcal{D}(X) = \mathcal{D}(\text{Cl}(X)) \quad (19)$$

We have a natural graded algebra inclusion of $R(X)$ into $\mathcal{D}(X)$ as the space of order zero differential operators; so $R_p(X) \subset \mathcal{D}_p^0(X)$. Moreover $R(X)$ is a maximal commutative subalgebra of $\mathcal{D}(X)$ living in non-negative degrees. A natural question is to find other maximal commutative subalgebras, and to see what degrees they live in.

To figure out what we might expect to find, let us consider the simplest example. This occurs when $V = \mathbb{C}^N$ is the standard representation of $G = GL(N, \mathbb{C})$ and so $X = \mathbb{C}^N - \{0\}$. Then $R(X) = R(\mathbb{C}^N)$ and so we get the Weyl algebra

$$\mathcal{D}(X) = \mathcal{D}(\mathbb{C}^N) = \mathbb{C}[v_1, \dots, v_n, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_n}] \quad (20)$$

In addition to $R(X) = \mathbb{C}[v_1, \dots, v_n]$, we have a second “obvious” maximal commutative subalgebra of $\mathcal{D}(X)$, namely the algebra

$$\mathcal{A} = \mathbb{C}[\frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_n}] \quad (21)$$

of constant coefficient differential operators. \mathcal{A} lives in non-positive degrees. The assignment $v_i \mapsto \frac{\partial}{\partial v_i}$ defines a graded algebra isomorphism $R(X) \rightarrow \mathcal{A}$. \mathcal{A} is generated by the space \mathcal{A}_{-1} of constant coefficient vector fields and $\mathcal{A}_{-p} \simeq V^{\boxtimes p}$ as G -representations.

We see no obvious way to generalize the construction of $L = \mathcal{A}_{-1}$ in the example above to the case where V is arbitrary. However, if one has such a space L then we can prove the following. This abstract result gives us the commutativity of the concrete exotic differential operators we construct later in Section 4.

Proposition 2.2.1 *Let $X \subset V$ and $X^* \subset V^*$ be the G -orbits of highest weight vectors. Suppose we have a finite-dimensional G -irreducible complex subspace $L \subset \mathcal{D}_{-1}(X)$ which carries the G -representation V . Then the differential operators $D \in L$ all commute and hence generate a commutative subalgebra $\mathcal{A} \subset \mathcal{D}(X)$. Consequently $\mathcal{A} = \bigoplus_{p \in \mathbb{Z}_+} \mathcal{A}_{-p}$ is graded in non-positive degrees with $\mathcal{A}_0 = \mathbb{C}$ and $\mathcal{A}_{-1} = L$.*

\mathcal{A} is isomorphic to $R(X^)$. In fact, if we fix a G -linear isomorphism $V \rightarrow L$, $v \mapsto D_v$, then the map $R_1(X^*) \rightarrow \mathcal{A}_{-1}$, $f_v \mapsto D_v$, extends uniquely to a graded G -linear algebra isomorphism $R(X^*) \rightarrow \mathcal{A}$, $f \mapsto D_f$.*

Proof: Let us consider the complex G -linear map $\pi : \wedge^2 V \rightarrow \text{End } R(X)$ given by $\pi(u \wedge v) = [D_u, D_v]$. The image of π lies in the space $\text{End}_{[-2]} R(X)$ of endomorphisms which are homogeneous of degree -2 with respect to the Euler action. So π is just a collection of complex G -linear maps

$$\pi_p : \wedge^2 V \rightarrow \text{Hom}(R_p(X), R_{p-2}(X)) \quad (22)$$

Thus π_p is an intertwining operator between two G -representations, namely $\wedge^2 V$ and $\text{Hom}((V^*)^{\boxtimes p}, (V^*)^{\boxtimes (p-2)}) \simeq \text{Hom}(V^{\boxtimes (p-2)}, V^{\boxtimes p})$.

But it follows from highest weight theory that

$$\text{Hom}_G(\wedge^2 V, \text{Hom}(V^{\boxtimes (p-2)}, V^{\boxtimes p})) = 0. \quad (23)$$

Indeed, let ν be the highest weight of V so that $V^{\boxtimes k} \simeq V_{k\nu}$. Every irreducible G -representation occurring in $\wedge^2 V$ has highest weight $\lambda < 2\nu$ by Cartan product theory. On the other hand, suppose μ is the highest weight of some irreducible representation occurring in $\text{Hom}(V_{(p-2)\nu}, V_{p\nu})$. The theory of Parasarathy, Rao and Varadarajan for tensor product decompositions gives here that $\mu \geq -(p-2)\nu + p\nu = 2\nu$. Thus $\lambda = \mu$ is impossible. This proves (23).

Therefore $\pi_p = 0$ for all p and so $\pi = 0$. This proves the commutativity of the operators D_v . The chosen map $V \rightarrow L$ defines a surjective G -linear graded algebra homomorphism $S(V) \rightarrow \mathcal{A}$. The image of $S^p(V)$ is the graded component \mathcal{A}_{-p} . Clearly $\mathcal{A}_{-p} \neq 0$ for $p \geq 0$.

We claim that the two graded algebra homomorphisms $S(V) \rightarrow \mathcal{A}$ and $S(V) \rightarrow R(X^*)$ have the same kernel. Indeed, suppose a G -irreducible subspace $W \subset \mathcal{A}_{-p}$ carries the representation with highest weight μ . Then arguing as above, we find that $\mu \leq p\nu$ since W occurs in $S^p(V)$ and also $\mu \geq p\nu$ because each operator $A \in \mathcal{A}_{-p}$ is a collection of maps $A_j : R_j(X) \rightarrow R_{j-p}(X)$. This forces $\mu = p\nu$. But the representation $V_{p\nu}$ occurs exactly once in $S^p V$, as the Cartan piece $V^{\boxtimes p}$. We conclude that the space $V^{\boxtimes p}$ maps isomorphically to \mathcal{A}_{-p} . The claim now follows. \square

Remark 2.2.2 (i) Proposition 2.2.1 was established in [B-K2, Theorem 3.10] for a restricted class of cases (\mathfrak{g}, V) . We have essentially just rewritten the proof given there in our more general situation.

(ii) Let $I \subset S(V)$ be the graded ideal of polynomial functions vanishing on X^* . Then we have a direct sum $S^p(V) = V^{\boxtimes p} \oplus I_p$. An important property of X^* is that I is generated by its degree two piece I_2 . This is a result of Kostant; see [Gar] for a write-up. This gives another way to prove the second paragraph of Proposition 2.2.1.

(iii) Suppose L is as in Proposition 2.2.1 but L carries the representation V^* instead of V . Then we can show by a similar argument that $V \simeq V^*$ and so the conclusions of the Proposition still follow.

Proposition 2.2.1 leads us to pose the question: in what generality does the space L exist? In [B-K2], L was constructed for a restricted set of cases of highest weight orbits (associated to complex Hermitian symmetric pairs). The main result of this paper is to construct L for the minimal nilpotent orbit \mathcal{O} in a classical complex simple Lie algebra different from $\mathfrak{sp}(2n, \mathbb{C})$.

Our final result of this subsection says, under a mild hypothesis, that the commutative subalgebra \mathcal{A} in Proposition 2.2.1 is maximal and the operators D_f define a non-degenerate inner product on $R(X)$. We will see later that our operators on \mathcal{O} satisfy the hypothesis and the resulting inner product is positive definite.

Let $U \subset G$ be a maximal compact subgroup of G . Then V admits a U -invariant positive-definite Hermitian inner product $(\cdot | \cdot)$ (unique up to a positive real scalar). Let \overline{V} denote the complex conjugate vector space to V . Then the complex conjugation map $V \rightarrow \overline{V}$, $v \mapsto \overline{v}$, is \mathbb{C} -anti-linear. Each vector $\overline{v} \in \overline{V}$ defines a \mathbb{C} -linear functional $b_{\overline{v}}$ on V by $b_{\overline{v}}(u) = (u | v)$. This gives a U -equivariant identification $\overline{V} \rightarrow V^*$, $\overline{v} \mapsto b_{\overline{v}}$, of complex vector spaces; we will use this identification freely from now on. This induces a graded \mathbb{C} -algebra identification $\overline{R(X)} = R(X^*)$. Then we get a graded \mathbb{C} -anti-linear algebra isomorphism $R(X) \rightarrow R(X^*)$, $g \mapsto \overline{g}$, defined in degree 1 by $f_{\overline{v}} \mapsto \overline{f_v} = f_v$.

Proposition 2.2.3 *Suppose we are in the situation of Proposition 2.2.1. Let us assume that for each positive integer p there exists an operator $D \in L$ such that D is non-zero on $R_p(X)$. Then*

- (i) The algebra \mathcal{A} , like $R(X)$, is a maximal commutative subalgebra of $\mathcal{D}(X)$.
- (ii) $R(X)$ admits a unique U -invariant non-degenerate Hermitian inner product $(\cdot|\cdot)$, such that $(1|1) = 1$ and, for all $v \in V$, the operators $f_{\bar{v}}$ and D_v are adjoint on $R(X)$. Then for any $f, g \in R(X)$ we have

$$(f|g) = \text{constant term of } D_{\bar{g}}(f) \quad (24)$$

Each $h \in R(X)$ can be uniquely written as a finite sum $h = \sum_{p \in \mathbb{Z}_+} h_p$ where $h_p \in R_p(X)$. Then, in view of (17), we call h_0 the *constant term* of h .

Proof: We see that (ii) implies (i) since $R(X)$ is a maximal commutative subalgebra of $\mathcal{D}(X)$. To prove (ii), we need to construct $B = (\cdot|\cdot)$.

We start by taking any U -invariant positive-definite Hermitian inner product Q on $R(X)$ such that $Q(1,1) = 1$. Clearly Q exists and $R_p(X)$ and $R_q(X)$ are Q -orthogonal if $p \neq q$ (since they carry different irreducible U -representations). Thus Q is given by a family, indexed by p , of inner products Q_p on $R_p(X)$. Similarly B must be given by a family B_p .

To start off, we put $B_0 = Q_0$. Now we proceed by induction and define B_{p+1} by the relation

$$B_{p+1}(h, f_{\bar{v}}g) = B_p(D_v(h), g) \quad (25)$$

where $g \in R_p(X)$ and $h \in R_{p+1}(X)$. This relation is exactly the condition that multiplication by $f_{\bar{v}}$ is adjoint to D_v .

We need to check that B_{p+1} is well-defined. Clearly the functions $f_{\bar{v}}g$ span $R_{p+1}(X)$. Also there exists a complex scalar c_{p+1} such that

$$c_{p+1}Q_{p+1}(h, f_{\bar{v}}g) = B_p(D_v(h), g) \quad (26)$$

To see this we observe that the two assignments $v \otimes \bar{g} \otimes h \mapsto Q_{p+1}(h, f_{\bar{v}}g)$ and $v \otimes \bar{g} \otimes h \mapsto B_p(D_v(h), g)$ both define U -equivariant complex linear maps

$$R_1(X) \otimes \overline{R_p(X)} \otimes R_{p+1}(X) \rightarrow \mathbb{C} \quad (27)$$

But the space of such maps is 1-dimensional – this follows using highest weight theory as in the proof of Proposition 2.2.1. So $B_{p+1} = c_{p+1}Q_{p+1}$. Thus B_{p+1} is well-defined. (This was the same proof given in [B-K2, Theorem 4.5].)

Our hypothesis that some D_v is non-zero on $R_{p+1}(X)$ ensures that $c_{p+1} \neq 0$. Hence B_{p+1} is non-degenerate. Finally (24) follows as $(f|g) = (D_{\bar{g}}(f)|1)$. \square

2.3 Complex minimal nilpotent orbits

From now on, we assume that G is a complex simple Lie group. Then the Lie algebra $\mathfrak{g} = \text{Lie } G$ is a complex simple Lie algebra and so the adjoint representation of G on \mathfrak{g} is irreducible. We put $V = \mathfrak{g}$ and identify $V \simeq V^*$ in an G -linear way using the Killing form.

The G -orbit X of highest weight vectors in \mathfrak{g} is now the minimal (non-zero) nilpotent orbit \mathcal{O} in \mathfrak{g} . Indeed, we can fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ where \mathfrak{h} is a complex Cartan subalgebra and \mathfrak{n}^+ and \mathfrak{n}^- are the spaces spanned by, respectively, the positive and negative root vectors. Let $\psi \in \mathfrak{h}^*$ be the highest (positive) root and let $H_\psi, Z_\psi, Z_{-\psi}$ be the corresponding root triple so that $H_\psi \in \mathfrak{h}$ is semisimple and $Z_{\pm\psi} \in \mathfrak{n}^\pm$ are nilpotent. Then Z_ψ is a highest root vector in \mathfrak{g} and also a highest weight vector in \mathfrak{g} . So $\mathfrak{g} = V_\psi$ and X is equal to the G -orbit \mathcal{O} of Z_ψ .

We choose a real Cartan involution σ of \mathfrak{g} such that \mathfrak{h} is σ -stable. Then the σ -fixed space in \mathfrak{g} is a maximal compact Lie subalgebra \mathfrak{u} . Then $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$ and σ is U -invariant and \mathbb{C} -anti-linear.

We can rescale the triple $(Z_\psi, H_\psi, Z_{-\psi})$ so that $\sigma(H_\psi) = -H_\psi$, $\sigma(Z_\psi) = -Z_{-\psi}$, $\sigma(Z_{-\psi}) = -Z_\psi$, and the bracket relations $[Z_\psi, Z_{-\psi}] = H_\psi$, $[H_\psi, Z_\psi] = 2Z_\psi$ and $[H_\psi, Z_{-\psi}] = -2Z_{-\psi}$ are preserved. Let $(\cdot, \cdot)_\mathfrak{g}$ be the complex Killing form on \mathfrak{g} rescaled so that $(Z_\psi, Z_{-\psi})_\mathfrak{g} = \frac{1}{2}$. We have a positive definite Hermitian inner product $(\cdot|\cdot)$ on \mathfrak{g} defined by $(u|v) = -(u, \sigma(v))_\mathfrak{g}$.

Let \mathfrak{g}_k be the k -eigenspace of $\text{ad } h$ on \mathfrak{g} where we set

$$h = H_\psi \tag{28}$$

Since h is semisimple, \mathfrak{g} decomposes into the direct sum of the eigenspaces \mathfrak{g}_k . It is well known that the eigenvalues of $\text{ad } H_\psi$ lie in $\{\pm 2, \pm 1, 0\}$ and $\mathfrak{g}_{\pm 2} = \mathbb{C}Z_{\pm\psi}$. Thus we get the decomposition

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \tag{29}$$

We have $\dim \mathfrak{g}_k = \dim \mathfrak{g}_{-k}$ and the spaces $\mathfrak{g}_{\pm 1}$ are even dimensional. We put $m = \frac{1}{2} \dim \mathfrak{g}_{\pm 1}$. Then (see, e.g., [B-K3])

$$\dim \mathcal{O} = 2m + 2 \tag{30}$$

We now put

$$x_\psi = Z_\psi, \quad x'_0 = H_\psi, \quad x_0 = Z_{-\psi} \tag{31}$$

Then

$$(x_0|x_0) = 1/2 \quad \text{and} \quad [x'_0, x_0] = -2x_0 \tag{32}$$

By [A-B2], we can find a complex basis $x_1, \dots, x_m, x'_1, \dots, x'_m$ of \mathfrak{g}_{-1} such that

$$\begin{aligned} (x_i | x_i) &= (x'_i | x'_i) = 1/2 \\ [x_i, x_j] &= [x'_i, x'_j] = 0 \quad \text{and} \quad [x'_i, x_j] = \delta_{ij} x_0 \end{aligned} \tag{33}$$

We have, as in (18), a G -linear isomorphism

$$\mathfrak{g} \rightarrow R_1(\mathcal{O}), \quad x \mapsto f_x \tag{34}$$

where $f_x(y) = (x, y)_{\mathfrak{g}}$. Also as in (11) there is an infinitesimal vector field action

$$\mathfrak{g} \rightarrow \mathfrak{Vect} \mathcal{O}, \quad x \rightarrow \eta^x \tag{35}$$

defined by differentiating the G -action on \mathcal{O} . Then $\eta^x(f_v) = f_{[x,v]}$. We have the complex algebra gradings

$$R(\mathcal{O}) = \bigoplus_{p \in \mathbb{Z}_+} R_p(\mathcal{O}) \quad \text{and} \quad \mathcal{D}(\mathcal{O}) = \bigoplus_{p \in \mathbb{Z}} \mathcal{D}_p(\mathcal{O}) \tag{36}$$

defined by the Euler \mathbb{C}^* -action as in (13), since $R_p(\mathcal{O}) = 0$ for $p < 0$.

All our constructions on \mathcal{O} thus far have been algebraic holomorphic. However now we introduce complex conjugation on \mathcal{O} into the picture. (See e.g. [B1, Appendix, A.5] for a discussion of complex conjugation on varieties.)

The map $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ is a \mathbb{C} -anti-linear involution of \mathfrak{g} preserving \mathcal{O} . It follows that σ , as well as $-\sigma$, is an anti-holomorphic involution of \mathcal{O} which is also real algebraic. We define *complex conjugation* on \mathcal{O} to be the U -invariant map

$$\nu : \mathcal{O} \rightarrow \mathcal{O}, \quad \nu(z) = -\sigma(z) \tag{37}$$

Now ν defines complex conjugation maps

$$R(\mathcal{O}) \rightarrow R(\mathcal{O}), \quad f \mapsto \bar{f} \quad \text{and} \quad \mathcal{D}(\mathcal{O}) \rightarrow \mathcal{D}(\mathcal{O}), \quad D \mapsto \bar{D} \tag{38}$$

where \bar{f} and \bar{D} are given by $\bar{f}(z) = \overline{f(\nu(z))}$ and $\bar{D}(f) = \overline{D(\bar{f})}$. These complex conjugation maps are \mathbb{C} -anti-linear \mathbb{R} -algebra homomorphisms; in particular $\overline{f_1 f_2} = \bar{f}_1 \bar{f}_2$ and $\overline{D_1 D_2} = \bar{D}_1 \bar{D}_2$.

Lemma 2.3.1

- (i) We have $\bar{E} = E$ and consequently complex conjugation preserves the Euler gradings of $R(\mathcal{O})$ and $\mathcal{D}(\mathcal{O})$ in (36).
- (ii) For $x \in \mathfrak{g}$, we have $\bar{f}_x = -f_{\sigma(x)}$ and $\bar{\eta}^x = \eta^{\sigma(x)}$.

Proof: (i) is clear since ν is \mathbb{C} -anti-linear. For (ii) we note that $[x, y] = [\sigma(x), \sigma(y)]$ and $(x, y)_{\mathfrak{g}} = \overline{(\sigma(x), \sigma(y))_{\mathfrak{g}}}$ for all $x, y \in \mathfrak{g}$ since σ is a anti-complex Lie algebra involution of \mathfrak{g} . Consequently

$$\overline{f_x}(z) = \overline{f_x(-\sigma(z))} = -\overline{(x, \sigma(z))_{\mathfrak{g}}} = -(\sigma(x), z)_{\mathfrak{g}} = -f_{\sigma(x)}(z)$$

This computes $\overline{f_x}$. Using this we find

$$\overline{\eta^x}(f_y) = -\overline{\eta^x(f_{\sigma(y)})} = -\overline{f_{[x, \sigma(y)]}} = f_{[\sigma(x), y]} = \eta^{\sigma(x)}(f_y)$$

But any vector field on \mathcal{O} is uniquely determined by its effect on the functions $f_y \in R_1(\mathcal{O})$. So this proves that $\overline{\eta^x} = \eta^{\sigma(x)}$. \square

3 Symbols and Quantization on \mathcal{O}

3.1 Set-up

If X is a complex algebraic manifold, then the holomorphic cotangent bundle $T^*X \rightarrow X$ is again a complex algebraic manifold. The order d symbol gives a linear map $\mathcal{D}^d(X) \rightarrow R^d(T^*X)$, $D \mapsto \Sigma_d(D)$, where $R^d(T^*X) \subset R(T^*X)$ is the subspace of functions which are homogeneous (polynomial) functions of degree d on the fibers of $T^*X \rightarrow X$. We have an algebra grading $R(T^*X) = \bigoplus_{d \in \mathbb{Z}_+} R^d(T^*X)$, which we call the *symbol grading*.

If $D \in \mathcal{D}(X)$ has order d , then $\Sigma_d(D)$ is the *principal symbol* of D , denoted by **symbol** D . See, e.g., [B-K2, Appendix] for a resume of some basic facts about differential operators and their symbols.

An algebraic holomorphic action of G on X induces a natural action of G on T^*X which is algebraic holomorphic symplectic. Then the canonical projection $T^*X \rightarrow X$ is G -equivariant; the induced action of G on T^*X is called the *canonical lift* of the G -action on X . The induced representation of G on $R(T^*X)$ is locally finite. The symbol grading is G -invariant and the principal symbol map is G -equivariant.

Let $\lambda \in R(T^*\mathcal{O})$ be the principal symbol of the Euler vector field E on \mathcal{O} , i.e.,

$$\lambda = \mathbf{symbol} E \tag{39}$$

Then λ is G -invariant. The canonical lift of the Euler \mathbb{C}^* -action on \mathcal{O} defines a G -invariant algebra grading $R(T^*\mathcal{O}) = \bigoplus_{p \in \mathbb{Z}} R_p(T^*\mathcal{O})$. Then

$$R_p(T^*\mathcal{O}) = \{\phi \in R(T^*\mathcal{O}) \mid \{\lambda, \phi\} = p\phi\} \tag{40}$$

We say that ϕ is *Euler homogeneous of degree p* if $\phi \in R_p(T^*\mathcal{O})$.

Now we get a G -invariant complex algebra bigrading

$$R(T^*\mathcal{O}) = \bigoplus_{p \in \mathbb{Z}, d \in \mathbb{Z}_+} R_p^d(T^*\mathcal{O}) \quad (41)$$

where $R_p^d(T^*\mathcal{O}) = R^d(T^*\mathcal{O}) \cap R_p(T^*\mathcal{O})$. The order d symbol mapping sends $\mathcal{D}_p^d(\mathcal{O})$ into $R_p^d(T^*\mathcal{O})$.

Let $\Phi^x \in R(T^*\mathcal{O})$ be the principal symbol of the vector field η^x , i.e.,

$$\Phi^x = \mathbf{symbol} \, \eta^x \quad \text{for all } x \in \mathfrak{g} \quad (42)$$

Then $\lambda, \Phi^x \in R_0^1(T^*\mathcal{O})$.

For $i = 0, \dots, m$ we set

$$f_i = f_{x_i} \quad \text{and} \quad f'_i = f_{x'_i} \quad (43)$$

We recall from [A-B2] that these $2m+2$ functions form a set of local étale coordinates on \mathcal{O} . We also put

$$f_\psi = f_{x_\psi} \quad (44)$$

We have a Zariski open dense algebraic submanifold \mathcal{O}^{reg} of \mathcal{O} defined by

$$\mathcal{O}^{reg} = \{z \in \mathcal{O} \mid f_0(z) \neq 0\} \quad (45)$$

This construction of \mathcal{O}^{reg} breaks the G -symmetry but not the infinitesimal \mathfrak{g} -symmetry by vector fields. By Proposition 2.1.1, $R(\mathcal{O})$ and $\mathcal{D}(\mathcal{O})$ sit inside $R(\mathcal{O}^{reg})$ and $\mathcal{D}(\mathcal{O}^{reg})$, respectively, as the spaces of \mathfrak{g} -finite vectors.

We introduce the following vector fields on \mathcal{O}^{reg} , where $i = 1, \dots, m$,

$$\Delta^{x_i} = \eta^{x_i} - \frac{f_i}{f_0} \eta^{x_0} \quad \text{and} \quad \Delta^{x'_i} = \eta^{x'_i} - \frac{f'_i}{f_0} \eta^{x_0} \quad (46)$$

and their principal symbols

$$\Theta^{x_i} = \Phi^{x_i} - \frac{f_i}{f_0} \Phi^{x_0} \quad \text{and} \quad \Theta^{x'_i} = \Phi^{x'_i} - \frac{f'_i}{f_0} \Phi^{x_0} \quad (47)$$

We define a polynomial $P(w_1, w'_1, \dots, w_m, w'_m)$ by

$$\frac{1}{24}(\text{ad } w)^4(x_\psi) = P(w_1, w'_1, \dots, w_m, w'_m)x_0 \quad (48)$$

where

$$w = \sum_{i=1}^m (w_i x_i + w'_i x'_i) \quad (49)$$

3.2 Main Results

We assume from now on that $\mathfrak{g} \neq \mathfrak{sp}(2n, \mathbb{C})$, $n \geq 1$. We recall

Theorem 3.2.1 [A-B2]

(i) *There exists a unique (up to scaling) non-zero G -linear map*

$$r : \mathfrak{g} \rightarrow R_{-1}^4(T^*\mathcal{O}), \quad x \mapsto r_x \quad (50)$$

So $r(\mathfrak{g})$ is the unique subspace in $R_{-1}^4(T^\mathcal{O})$ which is G -irreducible and carries the adjoint representation. A lowest weight vector $r_0 = r_{x_0}$ in $r(\mathfrak{g})$ is given by the formula*

$$r_0 = \frac{1}{f_0} \left(P(\Theta^{x_1}, \Theta^{x'_1}, \dots, \Theta^{x_m}, \Theta^{x'_m}) - \frac{1}{4} \lambda^2 (\Phi^{x_0})^2 \right) \quad (51)$$

(ii) *Suppose $p < 4$. Then there is no non-zero G -linear map $\mathfrak{g} \rightarrow R_{-1}^p(T^*\mathcal{O})$. I.e., $R_{-1}^p(T^*\mathcal{O})$ contains no copy of the adjoint representation.*

Remark 3.2.2 The formula (51) applies equally well when $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, $n \geq 1$. But then $P = 0$. The symbol $f_0^{-1}(\Phi^{x_0})^2$ easily quantizes to a differential operator on \mathcal{O} . See [A-B2].

Our main result is the G -equivariant quantization of these symbols r_x into differential operators D_x on \mathcal{O} in the cases where \mathfrak{g} is classical.

Theorem 3.2.3 *Assume \mathfrak{g} is a complex simple Lie algebra of classical type and $\mathfrak{g} \neq \mathfrak{sp}(2n, \mathbb{C})$, $n \geq 1$. Then there exists a unique differential operator D_0 on \mathcal{O} such that*

- (i) D_0 has order 4 and the principal symbol of D_0 is r_0 .
- (ii) D_0 is a lowest weight vector in a copy of \mathfrak{g} inside $\mathcal{D}(\mathcal{O})$.
- (iii) D_0 is an Euler homogeneous operator on $R(\mathcal{O})$ of degree -1 , i.e., $D_0 \in \mathcal{D}_{-1}(\mathcal{O})$. Moreover we construct D_0 as an explicit non-commutative polynomial in the vector fields $\Delta^{x_i}, \Delta^{x'_i}, \eta^{x_0}$, and E . We prove that

$$D_0(f_\psi^k) = \gamma(k) f_\psi^{k-1} \quad (52)$$

where $\gamma(k)$ is an explicit degree 4 polynomial in k with leading term k^4 and all coefficients non-negative.

Proof: The uniqueness follows from Theorem 3.2.1 as it says that r_0 is, up to scaling, the unique symbol $r \in R_{-1}^{\leq 4}(T^*\mathcal{O})$ such that r is a lowest weight vector of a copy of the adjoint representation. We prove the existence in the next section by case-by-case analysis for $\mathfrak{g} = \mathfrak{sl}(N, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{so}(N, \mathbb{C})$. See Propositions 4.2.3 and 4.3.3. \square

Property (ii) in Theorem 3.2.3 implies that the G -subrepresentation of $\mathcal{D}(\mathcal{O})$ generated by D_0 is irreducible and carries the adjoint representation. In fact we are getting an equivariant quantization in the following sense.

Corollary 3.2.4 *There exists a unique G -linear map*

$$\mathfrak{g} \rightarrow \mathcal{D}_{-1}^4(\mathcal{O}), \quad x \mapsto D_x \quad (53)$$

quantizing (50) in the sense that, for all $x \in \mathfrak{g}$,

$$\text{symbol } D_x = r_x \quad (54)$$

I.e., D_x has order 4 and the principal symbol of D_x is r_x . Then D_{x_0} is the operator D_0 constructed in Theorem 3.2.3.

We get two more corollaries to Theorem 3.2.3 because of Propositions 2.2.1 and 2.2.3. The first is immediate.

Corollary 3.2.5 *The differential operators D_x , $x \in \mathfrak{g}$, all commute and generate a maximal commutative subalgebra \mathcal{A} of $\mathcal{D}(\mathcal{O})$. Consequently $\mathcal{A} = \bigoplus_{p \in \mathbb{Z}_+} \mathcal{A}_{-p}$ is graded in non-positive degrees with $\mathcal{A}_0 = \mathbb{C}$ and $\mathcal{A}_{-1} = \{D_x \mid x \in \mathfrak{g}\}$.*

\mathcal{A} is isomorphic to $R(\mathcal{O})$. In fact, the mapping $f_x \mapsto D_x$ extends uniquely to a G -equivariant complex algebra isomorphism

$$R(\mathcal{O}) \rightarrow \mathcal{A}, \quad f \mapsto D_f \quad (55)$$

Corollary 3.2.6 *$R(\mathcal{O})$ admits a unique positive-definite Hermitian inner product $(\cdot|\cdot)$ invariant under the maximal compact subgroup U of G such that $(1|1) = 1$ and the operators \bar{f} and D_f are adjoint for all $f \in R(\mathcal{O})$. Then the formula (24) holds.*

Proof: Proposition 2.2.3 applies because (52) says that D_0 is non-zero on $R_p(X)$ for $p \geq 1$. This gives us a unique non-degenerate U -invariant Hermitian inner product $(\cdot|\cdot)$ on $R(\mathcal{O})$ such that $(1|1) = 1$ and \bar{f} and D_f are adjoint.

We need to check that $(\cdot|\cdot)$ is positive-definite on each space $R_p(\mathcal{O})$, $p \geq 1$. It suffices to check that $(f_\psi^p | f_\psi^p)$ is positive for $p \geq 1$. Lemma 2.3.1(ii) implies that

$$\overline{f_\psi} = f_0 \quad (56)$$

since $-\sigma(x_\psi) = x_0$ (see Subsection 3.1). So (24) and (52) give

$$(f_\psi^p | f_\psi^p) = D_{f_0^p}(f_\psi^p) = D_0^p(f_\psi^p) = \gamma(1) \cdots \gamma(p) \quad (57)$$

which is positive by Theorem 3.2.3. \square

Remark 3.2.7 We will show in [A-B3] that $\overline{D_x} = -D_{\sigma(x)}$.

3.3 Strategy for quantizing the symbol r_0

Here is our strategy for quantizing the symbol r_0 into the operator D_0 . We start from the formula (51) for the symbol r_0 . This says that $S = f_0 r_0$ where

$$S = P(\Theta^{x_1}, \Theta^{x'_1} \dots, \Theta^{x_m}, \Theta^{x'_m}) - \frac{1}{4} \lambda^2 (\Phi^{x_0})^2 \quad (58)$$

Our idea is to construct D_0 by first constructing a suitable quantization \mathbf{S} of S which is left divisible by f_0 , and then putting $D_0 = f_0^{-1} \mathbf{S}$.

Now S is an explicit polynomial (once we compute P) in the symbols $\Theta^{x_i}, \Theta^{x'_i}, \lambda, \Phi^{x_0}$ of the vector fields $\Delta^{x_i}, \Delta^{x'_i}, E, \eta^{x_0}$ on \mathcal{O}^{reg} . Hence S belongs to $R(T^* \mathcal{O}^{reg})$. In fact by Theorem 3.2.1, we know that S belongs to $R(T^* \mathcal{O})$.

The first step of our strategy is to quantize S into a differential operator \mathbf{S} on \mathcal{O}^{reg} using the philosophy of symmetrization from Weyl quantization. This means that we obtain \mathbf{S} by taking the expression for S , replacing each symbol $\Theta^{x_i}, \Theta^{x'_i}, \lambda, \Phi^{x_0}$ by the corresponding vector field, and symmetrizing to allow for any order ambiguities. E.g., $\Theta^{x'_i} \Theta^{x_i}$ quantizes to $\frac{1}{2}(\Delta^{x'_i} \Delta^{x_i} + \Delta^{x_i} \Delta^{x'_i})$. We will refer to this procedure simply as “Weyl quantization”. (This does not determine the quantization uniquely, as one can symmetrize the symbol $(\Theta^{x'_i} \Theta^{x_i})^2$ in different ways; we merely choose a convenient scheme.)

Thus we construct \mathbf{S} as a non-commutative polynomial in the vector fields $\Delta^{x_i}, \Delta^{x'_i}, E, \eta^{x_0}$ on \mathcal{O}^{reg} . We want to quantize S into \mathbf{S} in a G -equivariant way so that \mathbf{S} , like S , is a lowest weight vector of a copy of $V_{2\psi} \simeq \mathfrak{g}^{\boxtimes 2}$. It turns out to be too hard (and unnecessary) to verify all this right away, but we do construct \mathbf{S} so that the following properties are evident:

$$[\eta^z, \mathbf{S}] = 0 \text{ for all } z \in \mathfrak{g}_{neg} \quad \text{and} \quad [\eta^h, \mathbf{S}] = -4\mathbf{S} \quad (59)$$

where

$$\mathfrak{g}_{neg} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \quad (60)$$

The conditions in (59) go half-way towards proving that \mathbf{S} is a lowest weight vector of a copy of $\mathfrak{g}^{\boxtimes 2}$. Indeed, we have the following easy fact from the Cartan-Weyl theory of representations.

Lemma 3.3.1 *A vector v in a representation $\rho : \mathfrak{g} \rightarrow \text{End} W$ is a lowest weight vector of a copy of $V_{k\psi} \simeq \mathfrak{g}^{\boxtimes k}$ if and only if the following four conditions are satisfied:*

- (i) v is \mathfrak{g} -finite
- (ii) $\rho_h(v) = -2kv$
- (iii) $\rho_x(v) = 0$ for all $x \in \mathfrak{g}'_0$ where \mathfrak{g}'_0 is the orthogonal space to h in \mathfrak{g}_0
- (iv) $\rho_z(v) = 0$ for all $z \in \mathfrak{g}_{neg}$

The second step is to show that $f_0^{-1}\mathbf{S}$ belongs to $\mathcal{D}(\mathcal{O})$, (We skip the intermediate step of showing that \mathbf{S} belongs to $\mathcal{D}(\mathcal{O})$.) In fact, the multiplication operator f_0 commutes with all the vector fields $\Delta^{x_i}, \Delta^{x'_i}, E, \eta^{x_0}$, and so commutes with \mathbf{S} . Thus we need not distinguish between $f_0^{-1}\mathbf{S}$ and $\mathbf{S}f_0^{-1}$.

We accomplish the second step by implementing Proposition 3.3.4 below. Before giving that result, we first prove two more general Lemmas about extending differential operators from \mathcal{O}^{reg} to \mathcal{O} .

Lemma 3.3.2 *Suppose \mathbf{T} is a differential operator on \mathcal{O}^{reg} . Then \mathbf{T} extends to a differential operator on \mathcal{O} if and only if the operator $\mathbf{T} : R(\mathcal{O}^{reg}) \rightarrow R(\mathcal{O}^{reg})$ satisfies $\mathbf{T}(R(\mathcal{O})) \subset R(\mathcal{O})$.*

Proof: Immediate as \mathcal{O} is quasi-affine; see e.g., [B-K2, Appendix, A.4 and A.6]. \square

Lemma 3.3.3 *Suppose \mathbf{T} is a differential operator on \mathcal{O}^{reg} such that $[\eta^z, \mathbf{T}] = 0$ for all $z \in \mathfrak{g}_{neg}$. Then \mathbf{T} extends to a differential operator on \mathcal{O} if and only if $\mathbf{T}(f_\psi^k) \in R(\mathcal{O})$ for $k = 0, 1, 2, \dots$.*

Proof: The functions f_ψ^k , $k = 0, 1, 2, \dots$, form a complete set of highest weight vectors in $R(\mathcal{O})$ for the \mathfrak{g} -representation given by the vector fields η^x . Precisely, f_ψ^k generates $R_k(\mathcal{O})$ under the action of \mathfrak{n}^- so that $\mathcal{U}(\mathfrak{n}^-) \cdot f_\psi^k = R_k(\mathcal{O})$. We have $\mathfrak{n}^- \subset \mathfrak{g}_0 \oplus \mathfrak{g}_{neg}$ and also $\mathfrak{g}_0 \cdot f_\psi^k = \mathbb{C}f_\psi^k$ by Lemma 3.3.1(iii). It follows that

$$R_k(\mathcal{O}) = \mathcal{U}(\mathfrak{g}_{neg}) \cdot f_\psi^k \tag{61}$$

Now suppose $\mathbf{T}(f_\psi^k) \in R(\mathcal{O})$. Then, since \mathbf{T} commutes with η^z for all $z \in \mathfrak{g}_{neg}$, we get

$$\mathbf{T}(R_k(\mathcal{O})) = \mathbf{T}(\mathcal{U}(\mathfrak{g}_{neg}) \cdot f_\psi^k) = \mathcal{U}(\mathfrak{g}_{neg}) \cdot \mathbf{T}(f_\psi^k) \subset R(\mathcal{O})$$

Thus $\mathbf{T}(R(\mathcal{O})) \subset R(\mathcal{O})$ and so \mathbf{T} extends to a differential operator by Lemma 3.3.2. The converse is obvious. \square

To show that $f_0^{-1}\mathbf{S}$ belongs to $\mathcal{D}(\mathcal{O})$ we will use

Proposition 3.3.4 *Suppose \mathbf{S} is a differential operator on \mathcal{O}^{reg} such that \mathbf{S} is Euler homogeneous of degree 0 and \mathbf{S} satisfies (59). Then the following two properties are equivalent:*

- (i) $f_0^{-1}\mathbf{S} \in \mathcal{D}(\mathcal{O})$
- (ii) $\mathbf{S}(f_\psi^k) = \gamma_k f_0 f_\psi^{k-1}$ for $k = 0, 1, 2, \dots$ where $\gamma_0, \gamma_1, \dots$ are scalars.
Moreover (i) and (ii) imply
- (iii) $f_0^{-1}\mathbf{S}$ is a lowest weight vector of a copy of \mathfrak{g} in $\mathcal{D}_{-1}(\mathcal{O})$.

Proof: Put $\mathbf{T} = f_0^{-1}\mathbf{S}$. Then $[\eta^z, \mathbf{T}] = 0$ for all $z \in \mathfrak{g}_{neg}$ since η^z commutes with f_0 and \mathbf{S} . So Lemma 3.3.3 applies and says that $\mathbf{T} \in \mathcal{D}(\mathcal{O})$ if and only if $\mathbf{T}(f_\psi^k) \in R(\mathcal{O})$ for all $k = 0, 1, \dots$. Hence (ii) implies (i).

Conversely suppose (i) holds. Then $g = \mathbf{T}(f_\psi^k)$ lies in $R_{k-1}(\mathcal{O})$ and $\eta^h(g) = 2(k-1)g$. But it follows using (61) that f_ψ^{k-1} is, up to scaling, the unique eigenfunction in $R_{k-1}(\mathcal{O})$ of η^h with eigenvalue $2(k-1)$. So $g = \gamma_k f_\psi^{k-1}$. This proves (ii).

Finally we show that properties (i) and (ii) imply (iii). We can apply Lemma 3.3.1 where $W = \mathcal{D}(\mathcal{O})$, $v = \mathbf{T} = f_0^{-1}\mathbf{S}$ and $k = 1$. Now condition (i) in Lemma 3.3.1 is satisfied since the whole \mathfrak{g} -representation $\mathcal{D}(\mathcal{O})$ is locally finite. Also conditions (ii) and (iv) follow from (59). So we need to check (iii).

Let $x \in \mathfrak{g}'_0$. We want to show that the differential operator $[\eta^x, \mathbf{T}]$ on \mathcal{O} is zero. We can do this by showing that $[\eta^x, \mathbf{T}]$ annihilates $R(\mathcal{O})$. Now $\eta^x(f_\psi) = 0$ (use (29)) and so $\mathbf{T}(f_\psi^k) = \gamma_k f_\psi^{k-1}$ implies $[\eta^x, \mathbf{T}](f_\psi^k) = 0$. But also $[\eta^x, \mathbf{T}]$ commutes with the vector fields η^z , $z \in \mathfrak{g}_{neg}$. Indeed $[\eta^z, [\eta^x, \mathbf{T}]] = [\eta^{[z,x]}, \mathbf{T}] = [\eta^{z'}, \mathbf{T}] = 0$ as $z' = [z, x] \in \mathfrak{g}_{neg}$. So then $[\eta^x, \mathbf{T}]$ annihilates $R(\mathcal{O})$ because of (61). \square

In carrying out this plan for quantizing r_0 , we encountered a problem as we found no “Weyl quantization” of the term $\lambda^2(\Phi^{x_0})^2$ in (58) which produced an operator \mathbf{S} satisfying the condition $\mathbf{S}(f_\psi^k) = \gamma_k f_0 f_\psi^{k-1}$. (Such a Weyl quantization may or may not be possible here; our “Weyl quantization” of the first term is already not unique. Probably Weyl quantization fails to give the operators we want; cf. the quantization in [L-O].)

To get around this, we applied Weyl quantization to the first term in (58) after computing P explicitly. Then we “guessed” that $\lambda^2(\Phi^{x_0})^2$ should quantize into the operator

$$(E + c_1)(E + c_2)(\eta^{x_0})^2 \tag{62}$$

where c_1 and c_2 were unknown constants. There is no problem in this as

Lemma 3.3.5 *For any polynomial $q(E)$ in the Euler vector field, the operator $q(E)(\eta^{x_0})^2$ is a lowest weight vector of a copy of $V_{2\psi} \simeq \mathfrak{g}^{\boxtimes 2}$ in $\mathcal{D}(\mathcal{O})$.*

Thus we got a candidate for \mathbf{S} with unknown parameters c_1 and c_2 . We then calculated $\mathbf{S}(f_\psi^k)$ and “solved” for c_1 and c_2 by demanding that $\mathbf{S}(f_\psi^k)$ be a multiple of $f_0 f_\psi^{k-1}$. There was no guarantee a priori that solutions for c_1 and c_2 existed, but in fact we found a unique solution for the polynomial $(E + c_1)(E + c_2)$.

In our presentation below, we in fact use a general polynomial $q(E)$ in place of the particular polynomial $(E + c_1)(E + c_2)$. We show that the required relation $\mathbf{S}(f_\psi^k) = \gamma_k f_0 f_\psi^{k-1}$ forces $q(E)$ to be of the form $(E + c_1)(E + c_2)$ and we determine c_1 and c_2 .

In working this out, we do not yet have a unified treatment of all complex simple Lie algebras. Instead we treat only the cases where \mathfrak{g} is classical. We work out the polynomial P and the operator \mathbf{S} in Section 4, first for $\mathfrak{g} = \mathfrak{sl}(N, \mathbb{C})$ and then for $\mathfrak{g} = \mathfrak{so}(N, \mathbb{C})$.

We conjecture that Theorem 3.2.3 holds also for the five exceptional simple complex Lie algebras.

4 Explicit construction of D_0 for \mathfrak{g} classical

4.1 Coordinate representations of vector fields

Our aim is to construct D_0 as the quantization of the symbol r_0 . Our strategy, developed in Subsection 3.3, is to quantize the symbol $S = f_0 r_0$ into an operator \mathbf{S} which satisfies condition (ii) in Prop 3.3.4. To construct \mathbf{S} and compute $\mathbf{S}(f_\psi^k)$, we will simply work out everything in terms of our local coordinates f_0, f'_0, f_i, f'_i on \mathcal{O} from (43).

Fortunately, in proving Theorem 3.2.1 in [A-B2], we first found a formula for the highest weight vector $f_\psi \in R_1(\mathcal{O}) \simeq \mathfrak{g}$. This was:

Proposition 4.1.1 [A-B2] *The unique expression for the function $f_\psi \in R_1(\mathcal{O})$ in terms of our local coordinates f_0, f'_0, f_i, f'_i , $i = 1, \dots, m$, on \mathcal{O} is*

$$f_\psi = \frac{1}{f_0^3} P(f_1, f'_1, \dots, f_1, f'_1) - \frac{1}{4} \frac{(f'_0)^2}{f_0} \quad (63)$$

The unique expressions for the vector fields $\eta^{x_i}, \eta^{x'_i}, \eta^{x_0}$ in terms of our local coordinates f_i, f'_i, f_0, f'_0 are, where $i = 1, \dots, m$,

$$\eta^{x_i} = -f_0 \frac{\partial}{\partial f'_i} + f_i \frac{\partial}{\partial f'_0}, \quad \eta^{x'_i} = f_0 \frac{\partial}{\partial f_i} + f'_i \frac{\partial}{\partial f'_0}, \quad \eta^{x_0} = 2f_0 \frac{\partial}{\partial f'_0} \quad (64)$$

These follow as $\eta^x(f_y) = f_{[x,y]}$ for any $x, y \in \mathfrak{g}$. Then (46) gives

$$\Delta^{x_i} = -f_0 \frac{\partial}{\partial f'_i} - f_i \frac{\partial}{\partial f'_0}, \quad \Delta^{x'_i} = f_0 \frac{\partial}{\partial f_i} - f'_i \frac{\partial}{\partial f'_0} \quad (65)$$

The operators $\Delta^{x_i}, \Delta^{x'_i}, \eta^{x_0}$ also span a Heisenberg Lie algebra since

$$[\Delta^{x'_i}, \Delta^{x'_j}] = [\Delta^{x_i}, \Delta^{x_j}] = [\Delta^{x'_i}, \eta^{x_0}] = [\Delta^{x_i}, \eta^{x_0}] = 0, \quad [\Delta^{x_i}, \Delta^{x'_j}] = 2\delta_{ij}\eta^{x_0} \quad (66)$$

The next two facts are trivial to verify.

Lemma 4.1.2 *The operators $\Delta^{x_i}, \Delta^{x'_i}$ all commute with the vector fields η^z where z lies in the Heisenberg Lie algebra \mathfrak{g}_{neg} .*

Lemma 4.1.3 *We have the relations $\eta^h(f_0) = -2f_0$, $\eta^h(f'_0) = 0$, $\eta^h(f_i) = -f_i$, $\eta^h(f'_i) = -f'_i$ for $i = 1, \dots, m$, and $[\eta^h, \Delta^y] = -\Delta^y$ if $y \in \mathfrak{g}_{-1}$.*

Also, we have $\eta^{x_0}f_\psi = f_{[x_0, x_\psi]} = f_{-x'_0} = -f'_0$ and so using (64) we find

$$(\eta^{x_0})^2 f_\psi^k = [-2kf_0f_\psi + k(k-1)(f'_0)^2] f_\psi^{k-2} \quad (67)$$

To reduce the number of formulas, we introduce the superscript ϵ with $\epsilon = \pm 1$ where $\epsilon = 1$ indicates primed quantities while $\epsilon = -1$ indicates unprimed quantities. Thus $x_i^\epsilon = x'_i$ if $\epsilon = 1$ while $x_i^\epsilon = x_i$ if $\epsilon = -1$ and so on. Then (64) and (65) give

$$\eta^{x_i^\epsilon} = \epsilon f_0 \frac{\partial}{\partial f_i^{-\epsilon}} + f_i^\epsilon \frac{\partial}{\partial f'_0} \quad \Delta^{x_i^\epsilon} = \epsilon f_0 \frac{\partial}{\partial f_i^{-\epsilon}} - f_i^\epsilon \frac{\partial}{\partial f'_0} \quad (68)$$

4.2 The case $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, $n \geq 2$

In this case \mathfrak{g} is of type A_n . The minimal nilpotent orbit \mathcal{O} is

$$\mathcal{O} = \{X \in \mathfrak{g} \mid X^2 = 0, \text{rank}(X) = 1\} \quad (69)$$

Then $\dim \mathcal{O} = 2n$ and so

$$m = n - 1 \quad (70)$$

A standard basis of $\mathfrak{gl}(n+1, \mathbb{C})$ is given by the elementary matrices $E_{i,j}$, $0 \leq i, j \leq n$. We choose $x_\psi = E_{0,n}$, $x'_0 = E_{0,0} - E_{n,n}$, and $x_0 = E_{n,0}$. Then the normalized Killing form on \mathfrak{g} is given by $(X, X')_{\mathfrak{g}} = \frac{1}{2}(\text{Trace } XX')$. We take the Cartan involution σ to be $\sigma(X) = -X^*$.

The matrices $x'_p = E_{n,p}$ and $x_p = E_{p,0}$, $p = 1, \dots, m$, form a basis of \mathfrak{g}_{-1} and satisfy the conditions in (33). So we get

$$w = \sum_{p=1}^m (w_p E_{p,0} + w'_p E_{n,p}) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ w_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ w_m & 0 & \dots & 0 & 0 \\ 0 & w'_1 & \dots & w'_m & 0 \end{pmatrix} \quad (71)$$

Next we need to compute the polynomial P defined in (48). We have the standard operator identity $\text{Ad}(\exp tw) = \sum_{k=0}^{\infty} \frac{1}{k!} (\text{ad } tw)^k$ and so (48) gives

$$P(w_i, w'_i) \text{ is the coefficient of } t^4 x_0 \text{ in } (\text{Ad}(\exp tw)) \cdot x_\psi \quad (72)$$

We can easily compute this since

$$(\text{Ad}(\exp tw)) \cdot x_\psi = (\exp tw)x_\psi(\exp -tw) \quad (73)$$

where the right side is a matrix product. We find

$$\exp(tw) = I + tw + \frac{1}{2}t^2 \left(\sum_{p=1}^m w_p w'_p \right) E_{n,0} \quad (74)$$

since $w^2 = (\sum_{p=1}^m w_p w'_p) E_{n,0}$ and $w^3 = 0$. Then we get

$$P(w_1, w'_1, \dots, w_m, w'_m) = \frac{1}{4} \left(\sum_{p=1}^m w_p w'_p \right)^2 \quad (75)$$

Now plugging this expression for P into (63) and (51) we get

$$f_\psi = \frac{1}{4f_0^3} [a^2 - (f_0 f'_0)^2] \quad \text{and} \quad r_0 = \frac{1}{4f_0} [A^2 - \lambda^2 (\Phi^{x_0})^2] \quad (76)$$

where

$$a = \sum_{i=1}^m f_i f'_i \quad \text{and} \quad A = \sum_{i=1}^m \Theta^{x_i} \Theta^{x'_i} \quad (77)$$

To begin the process of quantizing $S = f_0 r_0$, we quantize A into the operator

$$\mathbf{A} = \frac{1}{2} \sum_{i=1}^m (\Delta^{x_i} \Delta^{x'_i} + \Delta^{x'_i} \Delta^{x_i}) \quad (78)$$

Lemma 4.2.1 *\mathbf{A} is a differential operator on \mathcal{O}^{reg} of order 2 with principal symbol A . \mathbf{A} is Euler homogeneous of degree 0. \mathbf{A} satisfies $[\eta^z, \mathbf{A}] = 0$ for all $z \in \mathfrak{g}_{neg}$ and $[\eta^h, \mathbf{A}] = -2\mathbf{A}$.*

Proof: The first statement is clear. The commutativity $[\eta^z, \mathbf{A}] = 0$ follows by Lemma 4.1.2 and the weight relation $[\eta^h, \mathbf{A}] = -2\mathbf{A}$ follows by Lemma 4.1.3. \square

By calculation, we find the following two formulas, where the answer is expressed in terms of our “ ϵ ” notation defined at the end of Subsection 4.1.

$$\Delta^{x_i^\epsilon}(a) = \epsilon f_0 f_i^\epsilon \quad \text{and} \quad \Delta^{x_i^\epsilon}(f_0 f'_0) = -f_0 f_i^\epsilon \quad (79)$$

So we get

$$\Delta^{x_i^\epsilon}(f_\psi) = \frac{\epsilon}{2} \frac{f_i^\epsilon(a + \epsilon f_0 f_0')}{f_0^2} \quad (80)$$

Next we determine $\mathbf{A}(f_\psi^k)$ starting from the identity

$$\mathbf{A}(f_\psi^k) = k(\mathbf{A}f_\psi)f_\psi^{k-1} + k(k-1)f_\psi^{k-2} \sum_{i=1}^m (\Delta^{x_i} f_\psi)(\Delta^{x_i'} f_\psi) \quad (81)$$

Using (80) and (76) we find $\Delta^{x_i}(f_\psi)\Delta^{x_i'}(f_\psi) = -f_i f_i' f_\psi / f_0$ and so

$$\sum_{i=1}^m (\Delta^{x_i} f_\psi)(\Delta^{x_i'} f_\psi) = -\frac{a f_\psi}{f_0} \quad (82)$$

Now we need the first term in (81). First (80) and (79) give

$$\Delta^{x_i^\epsilon} \Delta^{x_i^{-\epsilon}}(f_\psi) = \frac{-a + \epsilon f_0 f_0' - 2f_i f_i'}{2f_0} \quad (83)$$

and consequently

$$\frac{1}{2}(\Delta^{x_i} \Delta^{x_i'} + \Delta^{x_i'} \Delta^{x_i})(f_\psi) = -\frac{a + 2f_i f_i'}{2f_0} \quad (84)$$

Now summing over $i = 1, \dots, m$ we get

$$\mathbf{A}(f_\psi) = -\left(1 + \frac{m}{2}\right) \frac{a}{f_0} \quad (85)$$

Now substituting into (81) we get

$$\mathbf{A}(f_\psi^k) = -k\left(k + \frac{m}{2}\right) \frac{a f_\psi^{k-1}}{f_0} \quad (86)$$

We need to calculate $\mathbf{A}^2(f_\psi^k)$. Since \mathbf{A} commutes with multiplication by any power of f_0 , (86) gives

$$\mathbf{A}^2(f_\psi^k) = -k\left(k + \frac{m}{2}\right) f_0^{-1} \mathbf{A}(a f_\psi^{k-1}) \quad (87)$$

So the problem is to compute

$$\begin{aligned} \mathbf{A}(a f_\psi^{k-1}) &= (\mathbf{A}a) f_\psi^{k-1} + a(\mathbf{A}f_\psi^{k-1}) \\ &\quad + (k-1) f_\psi^{k-2} \sum_{i=1}^m (\Delta^{x_i} a)(\Delta^{x_i'} f_\psi) + (\Delta^{x_i} f_\psi)(\Delta^{x_i'} a) \end{aligned} \quad (88)$$

Easily $\mathbf{A}(a) = -mf_0^2$ and also (80) and (79) give

$$(\Delta^{x_i^\epsilon} a)(\Delta^{x_i^{-\epsilon}} f_\psi) = \frac{f_i f'_i}{2f_0} (-a + \epsilon f_0 f'_0) \quad (89)$$

So

$$\sum_{i=1}^m (\Delta^{x_i} a)(\Delta^{x'_i} f_\psi) + (\Delta^{x_i} f_\psi)(\Delta^{x'_i} a) = -\frac{a^2}{f_0} \quad (90)$$

Collecting all these terms we find

$$\mathbf{A}(af_\psi^{k-1}) = -mf_0^2 f_\psi^{k-1} - (k-1) \left(k + \frac{m}{2}\right) \frac{a^2}{f_0} f_\psi^{k-2} \quad (91)$$

Thus (87) gives

$$\mathbf{A}^2(f_\psi^k) = \left[mk \left(k + \frac{m}{2}\right) f_0 f_\psi + k(k-1) \left(k + \frac{m}{2}\right)^2 \frac{a^2}{f_0^2} \right] f_\psi^{k-2} \quad (92)$$

Our aim is to find a polynomial $q(E)$ in the Euler vector field E such that $\mathbf{S} = \frac{1}{4}(\mathbf{A}^2 - q(E)(\eta^{x_0})^2)$ satisfies $\mathbf{S}(f_\psi^k) = \alpha_k f_0 f_\psi^{k-1}$ for some scalars α_k . A priori, there is no guarantee that such an operator $q(E)$ exists. But fortunately (76) gives the relation

$$\frac{a^2}{f_0^2} - (f'_0)^2 = 4f_0 f_\psi \quad (93)$$

This is the unique linear relation in $R(\mathcal{O}^{reg})$ between the three functions $\frac{a^2}{f_0^2}$, $(f'_0)^2$ and $f_0 f_\psi$. We have $q(E)(f_\psi^k) = q(k)f_\psi^k$. Now comparing (92) with (67) we conclude that $q(E)$ exists if and only if $k(k-1)\left(k + \frac{m}{2}\right)^2 = q(k)k(k-1)$. Thus $q(k) = \left(k + \frac{m}{2}\right)^2$ is the unique solution. Then arithmetic gives a formula for α_k (which turns out to simplify very nicely). This proves

Lemma 4.2.2 *Let $\mathbf{S} = \frac{1}{4}(\mathbf{A}^2 - q(E)(\eta^{x_0})^2)$ where $q(E)$ is some polynomial in the Euler operator E . Then \mathbf{S} is a differential operator on \mathcal{O}^{reg} of order 4 with principal symbol S . \mathbf{S} is Euler homogeneous of degree 0 and satisfies the conditions in (59). The polynomial*

$$q(E) = \left(E + \frac{m}{2}\right)^2 \quad (94)$$

is the unique polynomial in E such that \mathbf{S} satisfies (ii) in Proposition 3.3.4. Then $\mathbf{S}(f_\psi^k) = \alpha_k f_0 f_\psi^{k-1}$ where

$$\alpha_k = k^2 \left(k + \frac{m-1}{2}\right) \left(k + \frac{m}{2}\right) \quad (95)$$

The polynomial α_k has leading term k^4 and all coefficients non-negative since $m \geq 1$. Proposition 3.3.4 now says that $f_0^{-1}\mathbf{S}$ is a lowest weight vector of a copy of \mathfrak{g} in $\mathcal{D}_{-1}(\mathcal{O})$. Thus we have proven

Proposition 4.2.3 *Let $\mathfrak{g} = \mathfrak{sl}(m+1, \mathbb{C})$, $m \geq 1$, and define \mathbf{A} by (78). The differential operator*

$$D_0 = \frac{1}{f_0}\mathbf{S} = \frac{1}{4f_0} [\mathbf{A}^2 - (E + \frac{m}{2})^2(\eta^{x_0})^2] \quad (96)$$

on \mathcal{O}^{reg} extends to an algebraic differential operator on \mathcal{O} . D_0 satisfies all three conditions in Theorem 3.2.3. We have

$$D_0(f_\psi^k) = \alpha_k f_\psi^{k-1} \quad (97)$$

where α_k is given by (95).

Remark 4.2.4 We can show that \mathbf{A} extends to a differential operator on \mathcal{O} .

4.3 The case $\mathfrak{g} = \mathfrak{so}(N, \mathbb{C})$, $N \geq 6$

Here \mathfrak{g} is of type D_n if $N = 2n$ is even or of type B_n if $N = 2n+1$ is odd. We unify our treatment of the two types by using the obvious inclusion $\mathfrak{so}(2n, \mathbb{C}) \subset \mathfrak{so}(2n+1, \mathbb{C})$. The “final answer” given in Proposition 4.3.3 below is stated in a uniform way.

To begin with, we introduce a convenient model for the complex Lie algebra $\mathfrak{g} = \mathfrak{so}(N, \mathbb{C})$. Let $u \cdot v$ be a symmetric non-degenerate complex bilinear form on \mathbb{C}^N . We have a complex linear Lie bracket on $\wedge^2 \mathbb{C}^N$ given by

$$[a \wedge b, c \wedge d] = (a \cdot c)b \wedge d + (b \cdot d)a \wedge c - (a \cdot d)b \wedge c - (b \cdot c)a \wedge d \quad (98)$$

Each vector $u \wedge v$ defines a skew-symmetric linear transformation $L_{u \wedge v}$ on \mathbb{C}^N by

$$L_{u \wedge v}(a) = (u \cdot a)v - (v \cdot a)u \quad (99)$$

Extending linearly, we obtain a natural complex Lie algebra isomorphism $\wedge^2 \mathbb{C}^N \rightarrow \mathfrak{so}(N, \mathbb{C})$, $z \mapsto L_z$. We use this to identify \mathfrak{g} with $\wedge^2 \mathbb{C}^N$.

Now the minimal nilpotent orbit \mathcal{O} is

$$\mathcal{O} = \{u \wedge v \in \mathfrak{g} = \wedge^2 \mathbb{C}^N \mid u \cdot u = u \cdot v = v \cdot v = 0, \quad u \wedge v \neq 0\} \quad (100)$$

The complex dimension of \mathcal{O} is $\dim \mathcal{O} = 2N - 6$, and so

$$m = N - 4 \quad (101)$$

Thus $m = 2n - 4$ if \mathfrak{g} has type D_n , while $m = 2n - 3$ if \mathfrak{g} has type B_n .

We next fix a convenient basis of \mathbb{C}^N . If $N = 2n + 1$, we let v_0 denote a vector in \mathbb{C}^N such that $v_0 \cdot v_0 = 1$; if $N = 2n$, we set $v_0 = 0$. Then we have a direct sum decomposition $\mathbb{C}^N = V \oplus \mathbb{C}v_0$ where V is the orthogonal space to v_0 . We fix a basis $v_1, v'_1, \dots, v_n, v'_n$ of V such that $v_i \cdot v_j = v'_i \cdot v'_j = 0$, and $v_i \cdot v'_j = \delta_{ij}$. This basis of V , together with v_0 if N is odd, is a basis of \mathbb{C}^N .

Now we choose

$$x_0 = -v_{n-1} \wedge v_n, \quad x_\psi = v'_{n-1} \wedge v'_n, \quad x'_0 = v_{n-1} \wedge v'_{n-1} + v_n \wedge v'_n \quad (102)$$

The normalized Killing form on \mathfrak{g} is given by

$$(a \wedge b, c \wedge d)_{\mathfrak{g}} = \frac{1}{2} [(a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)] \quad (103)$$

We take the Cartan involution σ of \mathfrak{g} to be the one induced in the natural way from the \mathbb{C} -anti-linear involution $\sigma : \mathbb{C}^N \rightarrow \mathbb{C}^N$ such that $\sigma(v_i) = v'_i$ and $\sigma(v_0) = v_0$.

We can now construct a nice basis of \mathfrak{g}_{-1} . This basis is given by the set $v_i^\epsilon \wedge v_{n-1}, v_i^\epsilon \wedge v_n$ where $i = 1, \dots, n-2$, $\epsilon = \pm 1$, together with $v_0 \wedge v_{n-1}, v_0 \wedge v_n$ if N is odd. We label the basis vectors in the following way:

$$\tilde{x}_0 = v_0 \wedge v_{n-1}, \quad \tilde{x}'_0 = v_0 \wedge v_n \quad (104)$$

and for $i = 1, \dots, n-2$,

$$x_i = v_i \wedge v_{n-1}, \quad \tilde{x}_i = v'_i \wedge v_{n-1}, \quad x'_i = v'_i \wedge v_n, \quad \tilde{x}'_i = v_i \wedge v_n \quad (105)$$

The bracket relations among basis vectors are all zero except for the following ones

$$[x'_i, x_j] = [\tilde{x}'_i, \tilde{x}_j] = \delta_{ij} x_0, \quad [\tilde{x}'_0, \tilde{x}_0] = x_0 \quad (106)$$

where $i, j \in \{1, \dots, n-2\}$.

Referring back to Subsection 2.3, we see now that all the relations in (33) are satisfied if we take $x_1, \dots, x_{n-2}, \tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{n-2}$ and $x'_1, \dots, x'_{n-2}, \tilde{x}'_0, \tilde{x}'_1, \dots, \tilde{x}'_{n-2}$ to be, respectively, the lists x_1, \dots, x_m and x'_1, \dots, x'_m . (Here we ignore \tilde{x}_0 and \tilde{x}'_0 if N is even.) We have broken the sets x_1, \dots, x_m and x'_1, \dots, x'_m into these peculiar groups because this is necessary to write down P in the next step.

Now we put

$$w = \tilde{w}_0 \tilde{x}_0 + \tilde{w}'_0 \tilde{x}'_0 + \sum_{i=1}^{n-2} (w_i x_i + \tilde{w}_i \tilde{x}_i + w'_i x'_i + \tilde{w}'_i \tilde{x}'_i) \quad (107)$$

Then by writing out w in matrix form and computing as in (72) and (73) we find the following formula for P :

$$\begin{aligned} P(w_i, \tilde{w}_i, w'_i, \tilde{w}'_i, \tilde{w}_0, \tilde{w}'_0) = & \frac{1}{4} \left(\left(\sum_{i=1}^{n-2} w_i w'_i - \tilde{w}_i \tilde{w}'_i \right) - \tilde{w}_0 \tilde{w}'_0 \right)^2 \\ & + \left(\left(\sum_{i=1}^{n-2} w'_i \tilde{w}_i \right) + \frac{1}{2} \tilde{w}_0^2 \right) \left(\left(\sum_{i=1}^{n-2} w_i \tilde{w}'_i \right) - \frac{1}{2} (\tilde{w}'_0)^2 \right) \end{aligned} \quad (108)$$

Then plugging this expression for P into (63) we get

$$f_\psi = \frac{1}{4f_0^3} (a^2 + 4bc - (f_0 f'_0)^2). \quad (109)$$

where

$$\begin{aligned} a &= \left(\sum_{i=1}^{n-2} f_i f'_i - \tilde{f}_i \tilde{f}'_i \right) - \tilde{f}_0 \tilde{f}'_0 \\ b &= \left(\sum_{i=1}^{n-2} f_i \tilde{f}'_i \right) + \frac{1}{2} (\tilde{f}_0)^2 \\ c &= \left(\sum_{i=1}^{n-2} f'_i \tilde{f}_i \right) - \frac{1}{2} (\tilde{f}'_0)^2 \end{aligned} \quad (110)$$

By making the substitutions, where $i = 1, \dots, m$,

$$f_i^\epsilon \mapsto \Theta^{x_i^\epsilon}, \quad \tilde{f}_i^\epsilon \mapsto \Theta^{\tilde{x}_i^\epsilon}, \quad \tilde{f}_0^\epsilon \mapsto \Theta^{\tilde{x}_0^\epsilon}$$

we get symbols A, B, C corresponding to the functions a, b, c . Then plugging our expression for P into (51) we get

$$r_0 = \frac{1}{4f_0} (A^2 + 4BC - \lambda^2 (\Phi^{x_0})^2) \quad (111)$$

The Weyl quantization of these principal symbols A, B, C are the operators

$$\begin{aligned} \mathbf{A} = & \frac{1}{2} \left(\sum_{i=1}^{n-2} \Delta x_i \Delta x'_i + \Delta x'_i \Delta x_i - \Delta \tilde{x}_i \Delta \tilde{x}'_i - \Delta \tilde{x}'_i \Delta \tilde{x}_i \right) \\ & - \frac{1}{2} (\Delta \tilde{x}_0 \Delta \tilde{x}'_0 + \Delta \tilde{x}'_0 \Delta \tilde{x}_0) \end{aligned} \quad (112)$$

$$\mathbf{B} = \left(\sum_{i=1}^{n-2} \Delta x_i \Delta \tilde{x}'_i \right) + \frac{1}{2} (\Delta \tilde{x}_0)^2 \quad (113)$$

$$\mathbf{C} = \left(\sum_{i=1}^{n-2} \Delta \tilde{x}_i \Delta x'_i \right) - \frac{1}{2} (\Delta \tilde{x}'_0)^2 \quad (114)$$

Lemma 4.3.1 *\mathbf{A}, \mathbf{B} and \mathbf{C} are differential operators on \mathcal{O}^{reg} of order 2 with respective principal symbols A, B, C . \mathbf{A}, \mathbf{B} and \mathbf{C} are all Euler homogeneous of degree 0 and each one commutes with the vector fields η^z , $z \in \mathfrak{g}_{neg}$. We have $[\eta^h, \mathbf{A}] = -2\mathbf{A}$, $[\eta^h, \mathbf{B}] = -2\mathbf{B}$, and $[\eta^h, \mathbf{C}] = -2\mathbf{C}$.*

Proof: Same as for Lemma 4.2.1. \square

Now routine calculation results in the formulas

$$\mathbf{A}(f_\psi^k) = -t_k \frac{a}{f_0} f_\psi^{k-1} \quad \mathbf{B}(f_\psi^k) = -t_k \frac{b}{f_0} f_\psi^{k-1} \quad \mathbf{C}(f_\psi^k) = -t_k \frac{c}{f_0} f_\psi^{k-1} \quad (115)$$

where

$$t_k = k(k + \frac{m}{2} - 1) \quad (116)$$

Next more computation gives the following formulas, where we have set $r_k = k + \frac{m}{2} - 2$.

$$\mathbf{A}^2(f_\psi^k) = \left((k-1) \frac{a^2 + 4bc + r_k a^2}{f_0^2} + m f_0 \right) t_k f_\psi^{k-2} \quad (117)$$

$$2\mathbf{CB}(f_\psi^k) = \left((k-1) \frac{a^2 + 4bc + 2r_k bc + a f_0' f_0}{f_0^2} + m f_0 f_\psi \right) t_k f_\psi^{k-2} \quad (118)$$

$$2\mathbf{BC}(f_\psi^k) = \left((k-1) \frac{a^2 + 4bc + 2r_k bc - a f_0' f_0}{f_0^2} + m f_0 f_\psi \right) t_k f_\psi^{k-2} \quad (119)$$

Now combining the last three calculations we get

$$\begin{aligned} & (\mathbf{A}^2 + 2\mathbf{BC} + 2\mathbf{CB})(f_\psi^k) \\ &= \left[3m t_k f_0 f_\psi + (k-1)(k + \frac{m}{2} + 1) t_k \frac{a^2 + 4bc}{f_0^2} \right] f_\psi^{k-2} \end{aligned} \quad (120)$$

Our aim is again to find $q(E)$ so that we can build the operator \mathbf{S} with the desired properties. Proceeding as in the previous subsection, we compare (120) with (67). By (109) we have the relation

$$\frac{a^2 + 4bc}{f_0^2} - (f_0')^2 = 4f_0 f_\psi \quad (121)$$

This is the unique linear relation in $R(\mathcal{O}^{reg})$ between the functions $\frac{a^2 + 4bc}{f_0^2}$, $(f_0')^2$, and $f_0 f_\psi$. As in the previous subsection, we obtain

Lemma 4.3.2 *Let*

$$\mathbf{S} = \frac{1}{4} ((\mathbf{A}^2 + 2\mathbf{BC} + 2\mathbf{CB}) - q(E)(\eta^{x_0})^2) \quad (122)$$

where $q(E)$ is some polynomial in the Euler operator E . Then \mathbf{S} is a differential operator on \mathcal{O}^{reg} of order 4 with principal symbol S . \mathbf{S} is homogeneous of degree 0 and satisfies the conditions in (59). The polynomial

$$q(E) = (E + \frac{m}{2} + 1)(E + \frac{m}{2} - 1) \quad (123)$$

is the unique polynomial in E such that \mathbf{S} satisfies (iii) in Proposition 3.3.4. Then $\mathbf{S}(f_\psi^k) = \beta_k f_0 f_\psi^{k-1}$ where

$$\beta_k = k(k+1)(k + \frac{m}{2} - 1)(k + \frac{m}{2} - \frac{1}{2}) \quad (124)$$

The polynomial β_k has leading term k^4 and all coefficients non-negative since $m \geq 2$. Proposition 3.3.4 now says that $f_0^{-1}\mathbf{S}$ is a lowest weight vector of a copy of \mathfrak{g} in $\mathcal{D}_{-1}(\mathcal{O})$. Thus we have proven

Proposition 4.3.3 *Let $\mathfrak{g} = \mathfrak{so}(m+4, \mathbb{C})$, $m \geq 2$. The differential operator on \mathcal{O}^{reg}*

$$D_0 = \frac{1}{f_0} \mathbf{S} \quad (125)$$

defined by (122) and (123) extends to an algebraic differential operator on \mathcal{O} . D_0 satisfies all three conditions in Theorem 3.2.3. We have

$$D_0(f_\psi^k) = \beta_k f_\psi^{k-1} \quad (126)$$

where β_k is given by (124).

Remark 4.3.4 We can show that \mathbf{A} , \mathbf{B} and \mathbf{C} extend to differential operators on \mathcal{O} .

5 Relations to other work

We return to the more general situation of K -orbits on \mathfrak{p} discussed in the introduction. The invariant theory of the action of K on \mathfrak{p} was analyzed in [K-R].

Suppose \mathfrak{g} is a simple Lie algebra and \mathfrak{k} has non-trivial center. Then the action of the center of K defines a K -invariant splitting $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Let Y be a K -orbit inside \mathfrak{p}^+ . Then $\mathcal{D}(Y)$ has been studied for various cases of this sort in [L-Sm], [L-Sm-St],

[L-St] where the functions and vector fields on Y do not generate $\mathcal{D}(Y)$. They construct an extension of the natural infinitesimal action $\mathfrak{k} \rightarrow \mathfrak{Vect} Y$ to a Lie algebra homomorphism $\pi : \mathfrak{g} \rightarrow \mathcal{D}(Y)$ where \mathfrak{p}^+ acts by multiplication operators and \mathfrak{p}^- acts by commuting order 2 differential operators on Y which are Euler homogeneous of degree -1 .

In particular, in [L-Sm-St], Levasseur, Smith and Stafford analyzed cases where $Y = O_{\min} \cap \mathfrak{p}^+$ and O_{\min} is the minimal nilpotent orbit of \mathfrak{g} . They prove that the kernel of the algebra homomorphism $\mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{D}(Y)$ defined by their map π is the Joseph ideal. In ([L-St]) Levasseur and Stafford analyzed $\mathcal{D}(Y)$ where Y is any K -orbit in \mathfrak{p}^+ and \mathfrak{g} is classical. Using the theory of Howe pairs and the Oscillator representation, they construct π for Y “sufficiently small” and prove that the kernel of the corresponding algebra homomorphism $\mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{D}(Y)$ is a completely prime maximal ideal.

Now suppose \mathfrak{g} is simple and \mathfrak{k} has trivial center. Then \mathfrak{p} is irreducible as a \mathfrak{k} -module. In [B-K2], R. Brylinski and B. Kostant studied cases of this sort where $Y = O_{\min} \cap \mathfrak{p}$. They constructed a space L of commuting differential operators on Y such that L is G -irreducible and carries the K -representation \mathfrak{p} . The operators in L have order 4 and are Euler homogeneous of degree -1 . The highest weight vector in L was constructed as the quotient by a multiplication operator of a homogeneous quartic polynomial in certain commuting vector fields of the \mathfrak{k} -action.

Our methods are an extension of that construction. However, our quantization is considerably more subtle as the our principal symbol r_0 involves a homogeneous quartic polynomial in the symbols of *non-commuting* vector fields (which lie outside the \mathfrak{k} -action). The quantization then requires not only symmetrization but also the introduction of lower order correction terms. Theorem 3.2.3 says that these correction terms are uniquely determined.

In [L-O], Lecomte and Ovsienko show that the algebra of polynomial symbols on \mathbb{R}^n admits a unique quantization (involving non-obvious lower order terms) equivariant under the vector field action of $\mathfrak{sl}(n+1, \mathbb{R})$ on \mathbb{R}^n which arises by embedding \mathbb{R}^n as the “big cell” in \mathbb{RP}^n . It would be interesting to see if there is a \mathfrak{g} -equivariant quantization map $\mathcal{Q} : R(T^*\mathcal{O}) \rightarrow \mathcal{D}(O)$.

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